Sines and more Sines.

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Evaluate

$$\int_0^{\pi} \left(\frac{\sin(nx)}{\sin(x)}\right)^2 dx$$

where n is a positive integer.

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Discussion

Clearly, for $n > 1$,

$$\frac{\sin(nx)}{\sin(x)} = \frac{\sin((n-1)x)\cos(x) + \cos((n-1)x)\sin(x)}{\sin(x)}$$

thus

$$\frac{\sin(nx)}{\sin(x)} = \cos((n-1)x) + \frac{\cos(x)(\sin((n-2)x)\cos(x) + \cos((n-2)x)\sin(x))}{\sin(x)}$$

$$= \cos((n-1)x) + \cos((n-2)x)\cos(x) + \frac{\sin((n-2)x)\cos^2(x)}{\sin(x)}$$

$$= \cos((n-1)x) + \cos((n-2)x)\cos(x) - \sin((n-2)x)\sin(x) + \frac{\sin((n-2)x)}{\sin(x)}.$$

We finally have that

$$\frac{\sin(nx)}{\sin(x)} = 2\cos((n-1)x) + \frac{\sin((n-2)x)}{\sin(x)} \tag{1}$$

Let $I_n = \int_0^\pi (\sin(nx)/\sin(x))^2 dx$. Then

$$I_n = 4\int_0^\pi \cos^2((n-1)x)dx + 4\int_0^\pi \frac{\cos((n-1)x)\sin((n-2)x)}{\sin(x)}dx + \int_0^\pi \left(\frac{\sin((n-2)x)}{\sin(x)}\right)^2 dx$$

Since $2\cos((n-1)x)\sin((n-2)x) = \sin((2n-3)x) + \sin(-x)$ we get

$$I_n - I_{n-2} = 4\int_0^\pi \cos^2((n-1)x)dx - 2\int_0^\pi dx + 2\int_0^\pi \frac{\sin((2n-3)x)}{\sin(x)}dx$$

Thus

$$I_n - I_{n-2} = 2\int_0^\pi \frac{\sin((2n-3)x)}{\sin(x)}dx. \tag{2}$$

Next we will compute $J_n = \int_0^\pi \sin(nx)/\sin(x)dx$. Since

$$J_n = \int_0^\pi 2\cos((n-1)x) + \frac{\sin((n-2)x)}{\sin(x)}dx$$

Thus

$$J_n - J_{n-2} = \int_0^\pi 2\cos((n-1)x)dx$$

Hence $J_n = J_{n-2}$ for $n > 2$. Clearly $J_1 = \pi$ and $J_2 = 0$, we get that $J_{2k} = 0$ and $J_{2k+1} = \pi$ for all positive integers k.

Therefore

$$I_n - I_{n-2} = 2\pi$$

And $I_1 = 1$ and $I_2 = 2\pi$, thus

$I_{2k} = 2k\pi$ and $I_{2k+1} = (2k+1)\pi$.

About half of the solutions were variations of the solution above and the others solved the problem by expressing sine function using the exponential function.