Tangents.

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Evaluate

$$\sum_{r=0}^{n-2} 2^r \tan\left(\frac{\pi}{2^{n-r}}\right)$$

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Discussion

Let

$$f(x)=\sum_{r=0}^{n-2}\ln(\cos(x2^{r-n})). \tag{1}$$

Then

$$f(x)=\ln\left(\cos\left(\frac{x}{2^n}\right)\cos\left(\frac{x}{2^{n-1}}\right)\cdots\cos\left(\frac{x}{2^2}\right)\right)$$

It is known that

$$\cos\left(\frac{x}{2^n}\right)\cos\left(\frac{x}{2^{n-1}}\right)\cdots\cos\left(\frac{x}{2^2}\right)=\frac{\sin\left(\frac{x}{2}\right)}{2^{n-1}\sin\left(\frac{x}{2^n}\right)}$$

See the solution to October 2019 problem for the equation above. Now we have that

$$f(x)=\ln\left(\sin\left(\frac{x}{2}\right)\right)-\ln\left(\sin\left(\frac{x}{2^n}\right)\right)-\ln\left(2^{n-1}\right)$$

Thus

$$f'(x)=\frac{1}{2}\cot\left(\frac{x}{2}\right)-\frac{1}{2^n}\cot\left(\frac{x}{2^n}\right)$$

From (1) it follows that

$$f'(x)=-\frac{1}{2^n}\sum_{r=0}^{n-2}2^r\tan\left(\frac{x}{2^{n-r}}\right)$$

Now by letting $x=\pi$, in the two equations above we get that

$$\sum_{r=0}^{n-2}2^r\tan\left(\frac{\pi}{2^{n-r}}\right)=\cot\left(\frac{\pi}{2^n}\right)$$

All submitted solutions had a different solution using the identity $\tan(x)=\cot(x)-2\cot(2x)$.

There is a very interesting geometrical interpretation as well.