Find the Function

Submission deadline: February 28^{th} 2020

Let x be a real number. Find the function whose power series is

$$\frac{x^3}{3!} + \frac{x^9}{9!} + \frac{x^{15}}{15!} + \cdots$$

We did not receive any correct solutions.

Discussion: Let $f(x)=e^x-e^{-x}$. Then,

$$f(x)=2\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\frac{x^9}{9!}+\frac{x^{11}}{11!}+\frac{x^{13}}{13!}+\frac{x^{15}}{15!}+\frac{x^{17}}{17!}+\cdots\right)$$

Let $w=(-1+i\sqrt{3})/2$. Then $w^3=1$. Hence

$$f(wx)=2\left(wx+\frac{x^3}{3!}+\frac{w^2x^5}{5!}+\frac{wx^7}{7!}+\frac{x^9}{9!}+\frac{w^2x^{11}}{11!}+\frac{wx^{13}}{13!}+\frac{x^{15}}{15!}+\frac{w^2x^{17}}{17!}\cdots\right)$$

and

$$f(w^2x)=2\left(w^2x+\frac{x^3}{3!}+\frac{wx^5}{5!}+\frac{w^2x^7}{7!}+\frac{x^9}{9!}+\frac{wx^{11}}{11!}+\frac{w^2x^{13}}{13!}+\frac{x^{15}}{15!}+\frac{wx^{17}}{17!}\cdots\right)$$

It easily follows that $f(x)+f(wx)+f(w^2x)$ is equal to

$$2\cdot 3\left(\frac{x^3}{3!}+\frac{x^9}{9!}+\frac{x^{15}}{15!}+\cdots\right)+2(1+w+w^2)\left(x+\frac{x^5}{5!}+\frac{x^7}{7!}+\frac{x^{11}}{11!}+\frac{x^{13}}{13!}+\frac{x^{17}}{17!}+\cdots\right)$$

Since $1+w+w^2=0$, it is easily seen that

$$f(x)+f(wx)+f(w^2x)=6\left(\frac{x^3}{3!}+\frac{x^9}{9!}+\frac{x^{15}}{15!}+\cdots\right)$$

Since $e^{wx}=e^{-x/2}(\cos(\sqrt{3}x/2)+i\sin(\sqrt{3}x/2))$ and $e^{w^2x}=e^{-x/2}(\cos(\sqrt{3}x/2)-i\sin(\sqrt{3}x/2))$, it follows that

$$f(x)+f(wx)+f(w^2x)=e^x-e^{-x}+2\cos(\sqrt{3}x/2)(e^{-x/2}-e^{x/2})$$

which can be simplified to get

$$(e^{x/2}-e^{-x/2})((e^{x/2}+e^{-x/2})-2\cos(\sqrt{3}x/2))$$

Therefore,

$$\frac{x^3}{3!}+\frac{x^9}{9!}+\frac{x^{15}}{15!}+\cdots=\frac{2}{3}\sinh(x/2)(\cosh(x/2)-\cos(\sqrt{3}x/2)).$$