Seven

Submission deadline: October 31^{st} 2018

Find the number of positive integers x that is less than or equal to 10,000 such that $2^x - x^2$ is not divisible by 7.

The problem was solved (using some computer software) by

- Daram Varam, *American University of Sharjah, UAE.*

Discussion: Clearly

$$x = 7n + m, 0 \leq m \leq 6.$$

We will analyze the divisibility for each value of m.

Notice that $10,000 = 7 \times 1428 + 4$.

If 7 divides x, then it divides x^2 and hence 7 does not divide $2^x - x^2$.

$$\text{Thus there are 1428 values when } m = 0. \tag{1}$$

Now we look at $1 \leq m \leq 6$. It is easy to see that

$$2^x - x^2 = (2^{7n+m} - m^2) - 7(7n^2 + 2nm)$$

Thus, it is easy to see that 7 divides $2^x - x^2$ if and only if 7 divides $2^{7n+m} - m^2$. We further write

$$2^{7n+m} - m^2 = 2^{7n}(2^m - m^2) + m^2(2^{7n} - 1)$$

It is easy to see that 7 divides $2^m - m^2$ when $m = 2, 4, 5, 6$. And 7 does not divide $2^m - m^2$ when $m = 1, 3$. Thus, we need to look at the divisibility of $2^{7n} - 1$ by 7. If n is a multiple of 3, then $2^{7n} - 1$ has the factor $2^3 - 1$. When n is not a multiple of 3, it is easy to see that 7 does not divide $2^{7n} - 1$. For each $m = 2, 4, 5, 6$ there are 476 multiples of 3 under 10,000. Thus

$$\text{There are } 4 \times (1428 - 476) \text{ values when } m \text{ is } 2, 4, 5 \text{ or } 6. \tag{2}$$

A similar argument shows that

$$\text{There are } 2 \times 953 \text{ values for } m = 1 \text{ or } 2. \tag{3}$$

Combining the values in (1), (2) and (3) it follows that there are 7142 values.